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THE ETHOS MODEL

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1. INTRODUCTION

ETHOS is a model designed for simulating the effects of a particular set of government policy instruments on the short-run behaviour of the housing construction industry and the incremental employment thereby generated. (ETHOS stands for Employment Through Home Ownership Stimulation.) The instruments are cash grants to purchasers of new homes, interest-free loans to such purchasers, and subsidization to reduce mortgage interest rates. The model has been implemented for applications on micro-computing equipment. An important characteristic of the model is that it relates only to "cyclical" variations; it is not designed to determine the long-run growth paths of housing stocks, starts, and other relevant variables, but rather the (nonseasonal) variations around those paths. If one thinks of the long-run paths as being determined by demographic factors and life cycle decisions by households based on intertemporal allocations of lifetime wealth, ETHOS may be viewed as a model of some aspects of the institutional constraints on life cycle choices and how the constraints might be reduced by public policy. An additional point to emphasize is that ETHOS is not a forecasting model. It is a simulation model designed to function within an artificial (but realistic) environment. It operates on the assumption of "typical" housing cycles and cycles in general economic activity, and is intended to provide information about how such cycles might be modified by use of the specified policy instruments.

The layout of this report is as follows. We begin, in the next section, with a general description of the model and its main features. We then discuss in detail the specification and derivation of each of the several blocks of equations that constitute the model. (The complete model, in summary form, and the definitions of all variables are provided in appendices.) The report ends with a description of the implementation of the model for simulation analysis.

2. GENERAL FEATURES OF THE MODEL

We note below a number of important general features of ETHOS before turning to the details of model specification and the derivation of equations.

Simulation Applications: As noted, the model is designed for the simulation of housing cycles and employment effects. Simulations are carried out with and without the government program in effect. A series of cycles can be simulated, rather than just a single one, so that the time path of dynamic effects can be traced. The simulated cycles are artificial but based on observed historical cycles in housing construction and the general economy. (The 25-year period 1963-87 was examined for this purpose.) A user of the computer version of the model can alter the length, amplitude, and timing of the cycles for experimental purposes, although in other respects the cyclical patterns are fixed. All variables are determined at quarterly intervals and in seasonally adjusted form; the model abstracts entirely from seasonality. A set of "standard" values is assigned to the parameters of the model but a user is free to change these values in particular experiments. (If no changes are specified the default values are the standard ones.) Of parti-

cular interest, of course, are the parameters that define the government program. These include the cash grant, interest-free loan, and interest rate reduction that the program might offer, as well as the dates of initiation and termination of the program.

The Natural Path: In the absence of cyclical variations and any government program the system of interest is assumed to move through time along what we term its "natural path." In effect, the natural path is equated with long-run trend functions fitted to the individual variables. The deviations about the natural path values are what are modelled in ETHOS. The natural path values themselves are entirely exogenous to the model. (The term "natural path" is chosen by analogy with such terms as "the natural rate of unemployment" and "the natural rate of growth" in the lexicon of economics.)

Variables Expressed as Proportions: As much as possible the variables in ETHOS are expressed as proportions or ratios. The model requires only a small number of variables measured in actual units to give it a proper scale. Many of the variables are expressed as proportionate differences from natural path values; others are expressed as ratios to natural path values or to relevant totals, as appropriate. The functioning of the model is thus largely independent of the "size" of the housing industry and the economy at large.

State of the Economy: The general macro-economy is represented by only a small number of variables. These include, personal income, the rate of interest on conventional mortgages, and the overall rate of unemployment. Personal income and the interest rate are exogen-

ous. The unemployment rate is endogenous, inasmuch as it responds to employment created in the housing industry as a result of the government program. Cyclical variation in new housing prices is provided for also in the set of inputs by including a housing construction price index in the form of a ratio to natural path value. Feed-back effects from the general economy to the housing market are ignored within the model.

Program Financing: An important point to note is that the employment and other implications of the financing of the government program are ignored. In practice the program might be financed out of tax revenue, by borrowing, by cuts elsewhere in the government budget, or by some combination of these alternatives. In each case there would be employment implications, and these should be taken account of in an overall assessment of the employment effects of the program. However, it has not been possible to model the effects of alternative financing arrangements; a full model of the general economy would be required for that purpose. ETHOS is thus concerned only with expenditure effects.

The Behaviour of Renters: The renter population and its income distribution play a central role in the model. The government program of interest would be directed to the stimulation of new owner-type housing construction, and the resulting creation of jobs; it would not be designed to stimulate the construction of rental units, although it has obvious implications for the construction of such units. Now, in practice a new owner-type home might be purchased by a household already owning a home or by a household that

had been renting. More generally, there are in any period flows of households from renter-type to owner-type units, from existing owner-type to newly constructed owner-type, and so on. A key assumption in ETHOS is that the behaviour of renters is the critical factor in determining these flows, and hence the demand for new owner-type housing, either directly or indirectly. A renter household may purchase a new owner-type home or it may purchase an existing home vacated by an owner household that has just purchased a new home. And of course there may be a sequence of sales and purchases: an owner household may purchase a new home and sell its old one to another owner household; the latter may then sell its home to a third owner household; and so on. But if it is assumed that owners will make their purchases of new homes conditional on the sale of their old ones, as seems reasonable, then ultimately in the chain there must be a purchase by a renter household. It is in this sense that the behaviour of renter households determines the demand for new housing even when renters do not themselves purchase such housing. The effectiveness of the government program is thus dictated by the ability of renters to purchase homes. A corollary assumption is that the price or other purchase conditions for existing owner-type housing will adjust in such a way that purchases of existing homes will be as attractive to renters as purchases of newly constructed homes, even though only the latter offer direct program benefits.

Demand Orientation: The model can be described as demand-oriented. It is assumed that the housing market responds in such a way as to satisfy any new demand for housing created by the program,

either from new production or from the existing stock of vacant units. In judging this assumption it is important to keep in mind that only deviations from the natural path are of concern here; long-run supply responses are another matter altogether. In the context in which ETHOS is relevant, it may be taken that a government program of the type envisaged would be in effect only during slack periods in the construction industry, and that productive capacity would therefore always be available, although adjustments of production levels may not be instantaneous.

Gross Debt Service Ratio: The gross debt service ratio -- the annual ratio of mortgage payments to income -- is taken to be the target of the government policy instruments in the first instance. That is to say, the three instruments (grants, loans, interest rate subsidies) are all directed towards reducing the mortgage payments required for the purchase of a new home, and hence making it possible for households to purchase when they could not otherwise do so, or inducing households already in a position to purchase to do so now (when the industry is in a slack phase) rather than wait until later. The policy instruments combine with the income distribution in such a way as to alter the gross debt service ratios and hence the purchase probabilities for renter households in different ranges of the distribution.

With the foregoing general features in mind we turn now to the details of ETHOS. We consider in turn each of the blocks of equations that constitute the model, giving attention to the derivation of the equations as well as to the final form in which they appear in the model. The complete model is presented in a summary form in

Appendix A and the definitions of all variables in the model in Appendix B.

3. STARTS, STOCKS, AND VACANCIES

This first block of equations determines housing starts, the stock of housing (number of units), and vacancies, for each of the two categories of housing that are of interest, namely owner-type housing and renter-type housing. There are eight equations in total. Starts, stocks, and vacancies are determined in the form of proportionate differences from natural values. The owner and renter vacancy rates are determined as proportions of stocks.

The equations for starts are developed as follows. Let S_{ht} be the number of starts for category h in period t (h=1 for owner-type units, h=2 for renter-type units) and let $\bar{S}_{ht}$ be the corresponding number on the natural path. We then write

$$(1) \quad S_{ht} = \bar{\alpha}_{h4}\sum_{i=0}^{3}\bar{\alpha}_{hit}Q_{it} + (1-\bar{\alpha}_{h4})S_{h,t-1} \qquad (h=1,2)$$

$$(2) \quad \bar{S}_{ht} = \bar{\alpha}_{h4}\sum_{i=0}^{3}\bar{\alpha}_{hit}\bar{Q}_{it} + (1-\bar{\alpha}_{h4})\bar{S}_{h,t-1} \qquad (h=1,2)$$

Q_i is the number of renter households in group i, where i indicates the group's position in relation to ability to purchase and the government program to stimulate the demand for new housing: Q_0 is the group at the lower end of the income distribution that is unable to buy even with the program in existence; Q_1 is the group that is able to buy, but only because of the program; Q_2 is the group that could buy without the program but is able to take advantage of the program; and Q_3 is the group able to buy but not able to benefit under the program because the value of housing that they would purchase exceeds the program's upper limit. (If there were no upper limit

stipulated, Q_3 would be null.) $\bar{Q}_0$, $\bar{Q}_1$, $\bar{Q}_2$, and $\bar{Q}_3$ are the corresponding numbers of households that would be in the four groups if the system were in the natural state; however, since there is no government program in the natural state, $\bar{Q}_1 = \bar{Q}_2 = 0$. The $\bar{\alpha}$ parameters attaching to the Q and $\bar{Q}$ variables are propensities to demand new owner-type units (if h=1) or new renter-type units (if h=2): $\bar{\alpha}_{h3t}$ is the propensity for households unaffected by the program; $\bar{\alpha}_{h1t}$ and $\bar{\alpha}_{h2t}$ are propensities for the two groups that are eligible under the program; $\bar{\alpha}_{h0t}$ is the propensity for the group unable to purchase, with or without the program. If there is no program, all renter households are in either group 0 or group 3. The $\bar{\alpha}_{h4}$ parameters are constants representing adjustment lags: if $\bar{\alpha}_{h4} = 1$, adjustment to demand is instantaneous; if $0 < \bar{\alpha}_{h4} < 1$ there is a delay. As noted below, the notion of delayed response is relevant to the determination of vacancy rates within the model.

The interpretation of equation (1) is as follows: If $\bar{\alpha}_{h4}$ were 1, the number of new homes demanded would be equal to $\sum\bar{\alpha}_{hit}Q_{it}$. With $\bar{\alpha}_{h4}$ a positive fraction, the number of starts is a convex combination of the number of new homes demanded and the previous level of production. With a sustained level of $\sum\bar{\alpha}_{hit}Q_{it}$, the number of starts and the number of new homes demanded will converge but along the way to convergence there will always be some discrepancy. Equation (2) is interpreted in a similar fashion.

Let $\bar{Q} = Q_0 + Q_1 + Q_2 + Q_3$ be the total natural number of renter households. (Note that the population of renter households is defined by the number in existence in the natural state; when the system is off the natural path the total number is the same although

the distribution of the total among the four categories may be different.) We assume that $\bar{Q}/\bar{S}_h$ is constant for h=1,2 and that the rate of growth of the housing system along the natural path is g (hence $\bar{S}_{ht} = (1+g)\bar{S}_{h,t-1}$). Subtracting (2) from (1), dividing on both sides by $\bar{S}_h$, and rearranging terms we may then write

$$(3) \quad s_{ht} = \alpha_{h4}\sum_{i=0}^{3}\alpha_{hit}(q_{it}-\bar{q}_{it}) + \alpha_{h5}(s_{h,t-1}) \qquad (h=1,2)$$

where
$$s_{ht} = (S_{ht}-\bar{S}_{ht})/\bar{S}_{ht} \qquad (h=1,2)$$
$$q_{it} = Q_{it}/\bar{Q}_t \qquad (i=0,1,2,3)$$
$$\bar{q}_{it} = \bar{Q}_{it}/\bar{Q}_t \qquad (i=0,1,2,3)$$
$$\alpha_{hit} = \bar{\alpha}_{hit}(\bar{Q}/\bar{S}_h) \qquad (h=1,2;i=0,1,2,3)$$
$$\alpha_{h4} = \bar{\alpha}_{h4} \qquad (h=1,2)$$
$$\alpha_{h5} = (1-\alpha_{h4})(1+g)^{-1} \qquad (h=1,2)$$

Starts are thus determined by shifts in the proportions of renter households in the four groups, relative to the proportions on the natural path, and by the adjustment lags in the system. Note that $\bar{q}_1 = \bar{q}_2 = 0$.

The derivation of the equation to determine the stocks of housing units proceeds as follows. Let K_{ht} and $\bar{K}_{ht}$ be actual and natural stocks at the beginning of period t for owners (h=1) and renters (h=2). We then write

$$(4) \quad K_{ht} = \bar{\beta}_{h0}K_{h,t-1} + \sum_{j=1}^{3}\bar{\beta}_{hj}S_{h,t-j} \qquad (h=1,2)$$

$$(5) \quad \bar{K}_{ht} = \bar{\beta}_{h0}\bar{K}_{h,t-1} + \sum_{j=1}^{3}\bar{\beta}_{hj}\bar{S}_{h,t-j} \qquad (h=1,2)$$

$\bar{\beta}_{h0}$ represents the retention rate and $1-\bar{\beta}_{h0}$ the rate of discard or demolition, which is assumed constant. The $\bar{\beta}_{hj}$ coefficients represent the proportions of housing starts that are completed in 1, 2, or 3 periods. (The periods are quarters of a year; thus we assume

that all starts are completed within three quarters.) Assuming $\bar{S}_h/\bar{K}_h$ to be constant, subtracting (5) from (4), dividing on both sides by $\bar{K}_h$, and rearranging terms, we obtain

$$(6) \quad k_{ht} = \beta_{h0}k_{h,t-1} + \sum_{j=1}^{3}\beta_{hj}s_{h,t-j} \qquad (h=1,2)$$

where $k_{ht} = (K_{ht}-\bar{K}_{ht})/\bar{K}_{ht}$

$$\beta_{h0} = \bar{\beta}_{h0}(\bar{K}_{h,t-1}/\bar{K}_{ht})$$

$$\beta_{hj} = \bar{\beta}_{hj}\bar{S}_{h,t-j}/\bar{K}_{ht} \qquad (j=1,2,3)$$

Assuming again that the housing system is growing along its natural path at a rate g we may redefine the β_{hj} as

$$\beta_{h0} = \bar{\beta}_{h0}(1+g)^{-1}$$
$$\beta_{hj} = \bar{\beta}_{hj}(\bar{S}_h/\bar{K}_h)(1+g)^{-j} \qquad (j=1,2,3)$$

The manner in which vacancy rates respond to changes in demand is determined by the α_{h4} and α_{h5} parameters of equation (3). The number of starts will be equal to the number of new housing units demanded in sector h if $s_{ht} = s'_{ht}$, where s'_{ht} is defined by

$$(7) \quad s'_{ht} = \sum_{i=0}^{3}\alpha_{hit}(q_{it}-\bar{q}_{it}) \qquad (h=1,2)$$

The difference between units demanded and new units supplied is then equal to $(s'-s)\bar{S}$. If this difference is met from existing vacancies we may write

$$(8) \quad v_{ht} = v_{h,t-1} + (\bar{v}_{ht}-\bar{v}_{h,t-1}) - (s'_{h,t-1}-s_{h,t-1})\bar{S}_{h,t-1} \qquad (h=1,2)$$

Vacancies at the beginning of period t are thus equal to vacancies at the beginning of the previous period, plus natural growth, minus

the shortfall of new production relative to demand. (Note that we are assuming that prices of existing vacant units adjust to any government program benefits available to purchasers of new homes so that purchasers are in fact indifferent between new and existing homes. We are also assuming that the stock of vacant units is sufficient to satisfy any surplus demand.) A rearrangement of equation (8) yields

$$(9) \quad v_{ht} = \gamma_{h0} v_{h,t-1} - \gamma_{h1}(s'_{h,t-1} - s_{h,t-1}) \qquad (h=1,2)$$

where $\gamma_{h0} = (1+g)^{-1}$, $\gamma_{h1} = \left(\bar{s}_h/\bar{v}_h\right)(1+g)^{-1}$, and $\bar{s}_h/\bar{v}_h$ is assumed constant.

The final equation in the block represents a pair of identities that define the vacancy rate φ in terms of v, k, and the natural vacancy rate $\bar{\varphi}$:

$$(10) \quad \varphi_{ht} = \left((1+v_{ht})/(1+k_{ht})\right)\bar{\varphi}_h \qquad (h=1,2)$$

The starts, stocks, and vacancies block of the model thus consists of equations (3), (6), (9), and (10), and associated definitional relations. Given $(q_i - \bar{q}_i)$, which is determined elsewhere in the model, these equations determine s_h, k_h, v_h, and φ_h.

4. RENTER INCOME DISTRIBUTION

As noted above, the income distribution of renters plays a key role in the model in determining the effects of the government program. The distribution is represented by a function derived from the following logit equation:

$$(11) \quad \ln\left[F_0(\bar{y})/(1-F_0(\bar{y}))\right] = \delta_0 + \delta_1 \ln\bar{y}$$

$F_0(\bar{y})$ is a base-period cumulative distribution function for $\bar{y}$, which is the income of an individual household when the system is on the natural path. Manipulating the equation provides two explicit forms of the base-period distribution function:

$$(12) \quad F_0(\bar{y}) = \left[1+\exp(-(\delta_0+\delta_1 \ln \bar{y}))\right]^{-1} = \left[1+\exp(-\delta_0)(\bar{y})^{-\delta_1}\right]^{-1}$$

The first of the two forms makes it clear that the distribution function is logistic in the logarithm of $\bar{y}$. Equation (11) was estimated using Family Expenditure Survey data (see below) and was found to fit the data extremely well.

We note that the distribution function shifts through time as households move along the natural path. For purposes of the model it is assumed that all individual household natural incomes change in the same proportion, as represented by $\bar{Y}_t/\bar{Y}_0$, where $\bar{Y}_0$ is average natural income in the base period and $\bar{Y}_t$ is the corresponding income in period t. (In practice, $\bar{Y}_t/\bar{Y}_0$ is calculated from national accounts personal income data.) This allows us to express $F_t(\bar{y})$, the period t distribution function, as

$$(13) \quad F_t(\bar{y}) = F_0(\bar{y}\cdot\bar{Y}_0/\bar{Y}_t) = \left[1+\exp(-\delta_0)(\bar{y}\cdot\bar{Y}_0/\bar{Y}_t)^{-\delta_1}\right]^{-1}$$

In effect, multiplying $\bar{y}$ by $\bar{Y}_0/\bar{Y}_t$ adjusts current household income back to what it was in the base period so that points on $F_t(\cdot)$ can be matched with corresponding points on $F_0(\cdot)$. The density function corresponding to $F_t(\bar{y})$ can be expressed simply as

$$(14) \quad f_t(\bar{y}) = dF_t/d\bar{y}$$

Now $\bar{y}$ is household income <u>on the natural path</u>. In fact, households will experience income fluctuations as the system ex-

periences cycles. However, we assign households to income groups based on their natural incomes; when their actual current incomes change, they are thus still classified to the same natural income groups. (An alternative way of saying this is to say that they are assigned to income groups on the basis of their permanent incomes, and that transitory fluctuations are ignored for this purpose. The permanent/transitory dichotomy is, of course, familiar to economists in the context of the "permanent income hypothesis." For present purposes "permanent" income and "natural" income may be taken as equivalent.)

We assume that actual incomes of renter households vary in the same proportion as overall average income (for renters and owners combined) over the course of a cycle. Specifically, we write

$$(15) \quad y_t = \bar{y}_t (Y_t/\bar{Y}_t)^{\epsilon}$$

where $(Y/\bar{Y})_t$ is the ratio of average household income to its natural value in period t and ϵ is an elasticity parameter. If $\epsilon = 1$, renter incomes fluctuate exactly in proportion to overall average income. However, ϵ need not be equal to 1.

5. POTENTIAL PURCHASERS

The values of $(q-\bar{q})$ required for the four income groups by the starts equations are determined in this block of the model. The determination, based on the income distribution function, is as follows:

(16) $q_{0t} = F_t(z_{1t})$

(17) $q_{it} = F_t(z_{i+1,t}) - F_t(z_{it})$ (i=1,2)

(18) $q_{3t} = 1 - F_t(z_{3t})$

(19) $\bar{q}_{0t} = F_t(\bar{z}_{1t}) = F_t(\bar{z}_{2t}) = F_t(\bar{z}_{3t})$ (since $\bar{z}_{1t} = \bar{z}_{2t} = \bar{z}_{3t}$)

(20) $\bar{q}_{it} = 0$ (i=1,2)

(21) $\bar{q}_{3t} = 1 - F_t(\bar{z}_{3t})$

The z and $\bar{z}$ values are points on the distribution of natural incomes: the $\bar{z}$ values are points when the system is in fact in the natural state and the z values are points when the system is in its actual current state. For example, z_{2t} is the upper bound of the income group that would be able to purchase a new home in the current state only because of the government program, z_{1t} is the lower bound of that group, and q_{1t} is thus the proportion of renters in income group 1; $\bar{q}_{1t} = 0$ is the corresponding proportion when the system is in the natural state (with no program). The difference, $q_{1t} - \bar{q}_{1t}$, is transferred to equation (3) as an argument in the functions that determine starts of owner-type and renter-type units. Note that $\bar{z}_{1t} = \bar{z}_{2t} = \bar{z}_{3t}$ because of the absence of any government program in the natural state. Note also that the lower bound of group 0 is zero, so that $F_t(z_{0t}) = F_t(\bar{z}_{0t}) = 0$, and that the upper bound of group 3 is infinity, so that $F_t(z_{4t}) = F_t(\bar{z}_{4t}) = 1$.

6. PROPENSITIES TO PURCHASE

The parameters attaching to the $q-\bar{q}$ terms in equation (3) represent propensities to demand housing units. It can be assumed

that benefits available under the government program will increase the propensities, and that the increases will be greater, the greater are the benefits. The average money value of the benefits available to a household in group 1 or group 2 is calculated in a subsequent block of the model. We assume that for groups 1 and 2 the propensity to purchase a home can be represented by an equation of the form

$$(22) \quad \alpha_{1it} = \alpha_{13}\left(1+n_{it}/MV(w)_{it}\right)^{\theta_i} \qquad (i=1,2)$$

where n_i is the average benefit to a member of group i and $MV(w)_i$ is the mean value of home purchases by members of that group. We assume that the base propensity is the same in groups 1 and 2 as in group 3, so that $\alpha_{11} = \alpha_{12} = \alpha_{13}$ when there is no government program (n=0); a positive value of n then raises the propensity above its base level, in accordance with the elasticity parameter θ. (α_{13} is taken to be constant over time: $\alpha_{13t} = \alpha_{13}$ for all t. Also, $\alpha_{10} = 0$.) It is convenient to define q_{11}, q_{12}, and q_{13} separately rather than lump them together when the government program is not in effect; this has no consequence for final results since in this case $n = 0$ and $\alpha_{11} = \alpha_{12} = \alpha_{13}$.

Changes in the demand for rental units may be viewed as the obverse of changes in the demand for owner-type units. In fact, therefore, it is not necessary to specify the α_2 coefficients; we can simply write

$$(23) \quad s'_{2t} = -\zeta s'_{1t}$$

where ζ is a parameter that determines the rates of reduction of demand for rental housing per unit increase in demand for owner-type

housing. If the rate is one rental unit less for every increase of one owner-type unit it can be shown that $\zeta = \bar{S}_2/\bar{S}_1$. However, if there is some degree of "doubling up" by households living in rental units the value of ζ may be less than $\bar{S}_2/\bar{S}_1$. We write $\zeta = \bar{\zeta}(\bar{S}_2/\bar{S}_1)$ and set $\bar{\zeta} = 1$ as a standard value. A user of the model is able to adjust $\bar{\zeta}$ to see how sensitive the simulation results are to changes in the assumed rate of exchange. With s'_{2t} known, s_{2t} can be calculated from equation (3) with the summation term on the right side of the equation replaced by s'_{2t}.

7. HOME VALUE

It is assumed for purposes of ETHOS that at any given time there is an exact correspondence between the natural income of a home purchaser and the natural value of the home purchased. It is assumed also that as income grows along the natural path the ratio of home value to income remains constant. (The latter assumption ensures nondegenerate steady-state growth, it may be noted.) In the chosen base period, the relationship between natural home value, $\bar{w}$, and natural household income, $\bar{y}$, is specified to be of the form

$$(24) \quad \bar{w}_0 = \eta_0 + \eta_1 \bar{y}_0$$

where η_0 and η_1 are parameters. If $\bar{w}/\bar{y}$ remains constant through time and all incomes increase at the same rate along the natural path, as assumed, we may then write

$$(25) \quad \bar{w}_t = (\eta_0 + \eta_1 \bar{y}_0)(\bar{y}_t/\bar{y}_0) = \eta_0(\bar{y}_t/\bar{y}_0) + \eta_1 \bar{y}_t$$

Note that equation (24) implies cross-section variation of $\bar{w}/\bar{y}$ with income <u>at a given time</u>, whereas (25) implies constancy of $\bar{w}/\bar{y}$ for a

given initial point in the income distribution as the distribution shifts through time.

Actual home values may depart from natural values because of price fluctuations. We therefore write

$$(26) \quad w_t = \bar{w}_t (P/\bar{P})_t$$

where w is actual home value and $(P/\bar{P})_t$ is the ratio of a general index of new home prices (P) to its natural level ($\bar{P}$) in period t.

8. FINANCING OF HOME PURCHASE

The purchase value of a new home is equal to the down payment, a, plus the amount borrowed under a standard mortgage contract, b, plus whatever funds the buyer can apply as a result of benefits under the government program. The benefits may take the form of one or more of a cash grant, an interest-free loan, and a subsidized mortgage interest rate. If c_1 is the amount available as a cash grant and c_2 the amount available as an interest-free loan, the new home value is related to sources of funds as follows:

$$(27) \quad w_t = a_t + b_t + \lambda_1 c_{1t} + \lambda_2 c_{2t}$$

λ_1 and λ_2 are parameters in the closed interval 0 to 1 representing the fractions of the grant and interest-free loan that (in effect) are applied to the home purchase. A grant or interest-free loan is conditional on the purchase of a new home and nominally must be applied to the purchase. However, in practice it may displace other funds that would have been used in the absence of the government program. If λ_1 or λ_2 is equal to zero there is full displacement, if equal to one there is no displacement, and if equal to a positive

fraction there is partial displacement. A user of the model can set the parameters at different levels to see how simulation results are affected.

We assume that standard practice by commercial lenders provides for a mortgage loan equal to a fraction ξ of home purchase value. The amount actually borrowed is reduced by the application of funds from the government program and so we write

$$(28) \quad b_t = \xi w_t - \lambda_1 c_{1t} - \lambda_2 c_{2t}$$

In addition, there is the possibility of a reduction of the quarterly mortgage interest rate ρ by an amount $\Delta\rho$ for some specified period of time. If T is the standard amortization period for a mortgage loan (say 100 quarters, or 25 years), we let T′ represent the period of subsidized interest rate. The interest subsidy is applied in such a way as to reduce the mortgage payments for period T′, the payments thereafter reverting to what they would have been without the subsidy for the balance of the amortization period, T − T′. Letting m_1 be the initial mortgage payment per quarter with the subsidy in effect, and m_2 the mortgage payment after the subsidy is terminated, we have

$$(29) \quad b_t = m_{1t}\sum_{1t} + m_{2t}\sum_{2t}$$

where $\sum_{1t} = \sum_{\tau=1}^{T'} (1+\rho_t-(\Delta\rho)_t)^{-\tau}$

$$\sum_{2t} = \sum_{\tau=T'+1}^{T} (1+\rho_t)^{-\tau}$$

The no-subsidy mortgage payment, m_2, is then determined as the payment that would have been required in the complete absence of

the subsidy:

(30) $m_{2t} = b_t / \sum_{3t}$

where $\sum_{3t} = \sum_{\tau=1}^{T} (1+\rho_t)^{-\tau}$

The gross debt service ratio plays an important role in the model. Indeed, the government program is designed with the aim of reducing this ratio in order to make possible new home purchases by households whose ratio would otherwise be too high to allow them to obtain conventional mortgage funds, as well as to encourage purchases by households who might otherwise postpone them. It is m_1, the initial quarterly mortgage payment, that determines the gross debt service ratio at the time of purchase. Noting that y is annual income, the annual gross debt service ratio, r, is defined by

(31) $r_t = 4m_{1t}/y_t$

9. INCOME BOUNDARY VALUES

The income points that define the upper and lower bounds of the four potential-purchaser groups are calculated in this block. Let $\hat{r}$ be the maximum allowable gross debt service ratio for purposes of obtaining a mortgage loan. If r is set equal to $\hat{r}$ on the left side of equation (31) and previous equations are used to substitute for m_1 and y on the right side of (31) we can obtain, by rearranging terms, the following equation for y^*, the minimum income level corresponding to $\hat{r}$:

(32) $y_t^* = A_t / B_t$

where $A_t = \$\eta_0 (\bar{Y}_t/\bar{Y}_0)(P/\bar{P})_t - (\lambda_1 c_{1t} + \lambda_2 c_{2t})$

$$B_t = (1/D_t)(\hat{r}/4)(Y/\bar{Y})_t^{\xi} - \xi\eta_1(P/\bar{P})_t$$

$$D_t = \left(1-\sum_{2t}/\sum_{3t}\right)/\sum_{1t}$$

Equation (32) is solved under three alternative sets of conditions to obtain values for z_1, z_2, and $\bar{z}_3$, as required by the Potential Purchasers block described previously:

$z_{1t} = y_t^*$ when all variables have actual values

$z_{2t} = y_t^*$ when $c_{1t} = c_{2t} = (\Delta\rho)_t = 0$

$\bar{z}_{3t} = y_t^*$ when $c_{1t} = c_{2t} = (\Delta\rho)_t = 0$, $(P/\bar{P})_t = (Y/\bar{Y})_t = 1$, and $\rho_t = \bar{\rho}$

$\bar{\rho}$ is the natural rate of interest. Note again that $\bar{z}_1 = \bar{z}_2 = \bar{z}_3$, so that $\bar{z}_1$ and $\bar{z}_2$ need not be calculated.

The value of z_3 is determined by the upper limit on the value of a new home for program eligibility. If the program specifies a limit $\hat{w}$ this may be substituted for w on the left side of equation (26). Substituting for $\bar{w}$ on the right side of (26) and rearranging terms gives the following equation for $\hat{y}$, the upper limit of $\bar{y}$ corresponding to the program's upper limit on home value:

(33) $$\hat{y}_t = \left[\hat{w}_t - \eta_0(\bar{Y}_t/\bar{Y}_0)(P/\bar{P})_t\right]/\eta_1(P/\bar{P})_t$$

We then calculate z_3 from

$z_{3t} = \hat{y}_t$ when all variables have actual values

If z_1 or z_2 exceeds z_3 it is set equal to z_3.

A final adjustment is made to the z and $\bar{z}$ values in order to bring them into accord with a reasonable independent estimate of the

minimum natural income required for a new house purchase in the absence of a government program. A minimum income value $\bar{Z}_0$ is specified for the base period, which in practice is the calendar year 1982. This value is moved forward in time using the relation

$$(34) \quad \bar{Z}_t = \bar{Z}_0(\bar{Y}_t/\bar{Y}_0)$$

The difference $\bar{Z}_t - \bar{z}_{3t}$ is then calculated and added to each of the z and $\bar{z}$ values. This implies that the recalculated $\bar{z}_{3t}$ is equal to $\bar{Z}_t$ and that all of the z values are shifted in such a way as to maintain the differences among them as originally calculated.

10. VALUE OF PROGRAM BENEFIT

The value of benefit available to a qualified new home purchaser is the combined present value of any cash grant, interest-free loan, and reduced stream of mortgage payments resulting from an interest rate subsidy. Letting n_i be this present value for a renter in group i, we write

$$(35) \quad n_{it} = c_{1t} + c_{2t}\left(1-(1+\rho_t')^{-T''}\right) + \left(MV(m_2-m_1)_{it}\right)\sum_{4t} \qquad (i=1,2)$$

where MV stands for mean value

$$\sum_{4t} = \sum_{\tau=1}^{T'}(1+\rho_t')^{-\tau}$$

$$m_2 - m_1 = \left[\xi(P/\bar{P})_t\left(\eta_0(\bar{y}_t/\bar{Y}_0)+\eta_1\bar{y}_t\right) - \left(\lambda_1 c_{1t}+\lambda_2 c_{2t}\right)\right]M_t$$

$$M_t = \left[1-\left(\sum_{3t}-\sum_{2t}\right)/\sum_{1t}\right]/\sum_{3t}$$

ρ' is the rate of interest at which the household discounts future returns or costs; ρ' may or may not be different from the mortgage interest rate ρ. T" is the period of the interest-free loan. The

expression for $m_2 - m_1$ is obtained by starting with equations (29) and (30), solving for $m_2 - m_1$, and then making a series of substitutions based on other equations specified previously. The integrals represented by the MV function are approximated numerically for purposes of the model.

11. PROGRAM PARTICIPATION

The number of new home buyers who participate in the government program is assumed proportional to the number of purchasers who are eligible. Specifically, we write

$$(36) \quad N_{1t} = \sigma_1\left(\alpha_{11t}q_{1t}/\textstyle\sum_{i=1}^{3}\alpha_{1it}q_{it}\right)(1+s_{1t})\bar{S}_{1t}$$

$$(37) \quad N_{2t} = \sigma_2\left(\alpha_{12t}q_{2t}/\textstyle\sum_{i=1}^{3}\alpha_{1it}q_{it}\right)(1+s_{1t})\bar{S}_{1t}$$

N_1 and N_2 are the numbers of program participants in groups 1 and 2. $(1+s_1)\bar{S}_1 = S_1$, the total number of starts of owner-type housing units. The term in the first pair of parentheses in each equation is taken to represent the fraction of new starts that are eligible in the given group. The parameters σ_1 and σ_2 are the proportions of eligible purchasers who actually participate. If $\sigma_1 = \sigma_2 = 1$ there is full participation by those eligible. However it is not uncommon in programs such as the one envisaged for there to be some slippage and a user of the model may wish to set σ_1 and σ_2 somewhat below one.

The total number of participants is $N_t = N_{1t} + N_{2t}$. This is the number of participants in quarter t. Of more interest is the number since the inception of the program. Letting AN_t stand for the accumulated number of participants up to and including quarter t we have the relation

(38) $AN_t = AN_{t-1} + N_t$

Given that the initial (pre-program) value is zero this relation can be applied recursively to determine AN in each period in which the program is in effect and over the life of the program as a whole.

12. PROGRAM COST

The total cost of the government program, excluding administrative cost, depends on the number of participants and the cost per participant associated with any cash grant, interest-free loan, or interest rate subsidy. The cost per participant is in fact the same as the benefit to an individual participating household, as derived in Section 10, except that the interest rate at which the government does the discounting calculation may differ from the individual household's (implicit) rate. With this in mind we write the following:

(39) $C_t = \sum_{i=1}^{2} N_{it}\left[c_{1t}+c_{2t}\left(1-(1+\rho_t'')^{-T''}\right)+\left(MV(m_2-m_1)_{it}\right)\sum_{5t}\right]$

where $\sum_{5t} = \sum_{\tau=1}^{T'}(1+\rho_t'')^{-\tau}$

ρ'' is the government's interest rate for discounting purposes.

13. EMPLOYMENT CREATION

The additional employment created by the government program is calculated on the basis of the difference between housing starts with the program in effect and starts in the absence of the program. $(\Delta E)_h$, the incremental employment (in person-quarters) for owner-type units (h=1) or renter-type units (h=2) is given by

$$(40)\quad (\Delta E)_{ht} = \sum_{j=0}^{2} \pi_{hj}\left(s_{h,t-j} - s^{*}_{h,t-j}\right)\bar{s}_{h,t-j} \qquad (h=1,2)$$

where s^{*} is the proportionate difference from natural starts when there is no program (i.e., the no-program value of s) and the π parameters represent person-quarters of employment per housing unit started. Provision is made for three kinds of employment effects, namely direct (on-site employment), indirect (employment created in other industries) and induced (employment resulting from increased demand in the economy at large because of the infusion of additional income). The overall employment-per-start parameters are thus written as

$$(41)\quad \pi_{hj} = \pi^{(1)}_{hj} + \pi^{(2)}_{hj} + \pi^{(3)}_{hj} \qquad (h=1,2;\ j=0,1,2)$$

where the superscripts denote direct, indirect, and induced effects, in that order. The j subscript indicates time lag since start to allow for the fact that housing units started as long ago as two quarters may still be under construction and generating employment.

Total employment generated is the sum of employment generated for owner-type units and employment generated for renter-type units. (In practice, an increase in one sector may be offset in part -- possibly large part -- by a decrease in the other.) We therefore have

$$(42)\quad (\Delta E)_{t} = (\Delta E)_{1t} + (\Delta E)_{2t}$$

In addition, the effect of employment creation on the overall national unemployment rate can be calculated under the (approximating) assumption that the size of the Canadian labour force is

independent of cyclical variation. Letting u^* be the unemployment rate in the absence of the government program and $\bar{L}$ be the labour force, the unemployment rate with the program in effect is then given by

$$(43) \quad u_t = u_t^* - (\Delta E)_t / \bar{L}_t$$

As noted previously, it is important to keep in mind that the model does not take account of employment effects associated with the financing of the government program, whether by taxation, borrowing, or substitution for other expenditures within the overall government budget. The effects of financing would have to be calculated outside of ETHOS.

14. NATURAL GROWTH RATES

This block of equations computes the natural levels of average personal income per household, labour force, and housing starts. Constant growth rates along the relevant portion of the natural path are assumed and the equations are as follows:

$$(44) \quad \bar{Y}_t = \bar{Y}_{t-1}\left(1+g(\bar{Y})\right)$$

$$(45) \quad \bar{L}_t = \bar{L}_{t-1}\left(1+g(\bar{L})\right)$$

$$(46) \quad \bar{S}_{ht} = \bar{S}_{h,t-1}\left(1+g(\bar{S})\right) \qquad (h=1,2)$$

The natural annual mortgage interest rate, $\bar{R}$, which is also required by the model, is assumed constant. The natural level of the new housing price index, $\bar{P}$, is never actually used within the model; only ratios $P/\bar{P}$ are used, so that no growth rate need be provided for $\bar{P}$.

15. DETERMINATION OF CYCLES

The cyclical patterns of $(Y/\bar{Y})$, $(P/\bar{P})$, R, and u* are obtained from sine functions representing "typical" or "stylized" time-series cycles. These functions are as follows:

$$(47) \quad (Y/\bar{Y})_t - 1 = d_1 \sin\left[(2\pi/p)(t-\omega_1)\right]$$

$$(48) \quad (P/\bar{P})_t - 1 = d_2 \sin\left[(2\pi/p)(t-\omega_2)\right]$$

$$(49) \quad (R-\bar{R})_t = d_3 \sin\left[(2\pi/p)(t-\omega_3)\right]$$

$$(50) \quad (u^*-\bar{u})_t = d_4 \sin\left[(2\pi/p)(t-\omega_4)\right]$$

where p is the period or length of cycle, the d parameters represent amplitudes, and the ω parameters represent time shifts in the cycles. The period is the same for all series. However, the amplitudes and shift parameters can differ. The shift parameters can be used to determine the desired phase points of the four cycles in the initial quarter of a simulation; by assigning different values to these parameters different lead-lag patterns among the series can be imposed. (Note that π in the above equations stands for the constant 3.1416; it should not be confused with the earlier use of π to represent employment generated per housing unit.)

16. QUARTERLY INTEREST RATES

The model uses quarterly interest rates. However, interest rates are quoted in annual form and conversion equations are thus required. For conventional mortgage rates the standard monthly form (with semi-annual compounding) is adapted to obtain the following

quarterly equations:

$$(51) \quad \rho_t = \left(1+R_t/2\right)^{1/2} - 1$$

$$(52) \quad \bar{\rho} = \left(1+\bar{R}/2\right)^{1/2} - 1$$

$$(53) \quad (\Delta\rho)_t = \left(1+R_t/2\right)^{1/2} - \left(1+\left(R_t-(\Delta R)_t\right)/2\right)^{1/2}$$

where R is the quoted annual rate, $\bar{R}$ is the corresponding natural annual rate, and ΔR is the reduction of the annual interest rate under the government program. $\Delta\rho$ is calculated as the difference between the quarterly rates with and without interest subsidy.

Values for ρ' and ρ'' are derived from equations similar to (51) with R replaced by R' and R'', the annual rates corresponding to the quarterly rates ρ' and ρ''.

17. IMPLEMENTATION OF THE MODEL

The model has been adapted for practical applications in the form of a FORTRAN program written to run on PC-compatible micro-computing equipment. The parameters of the model have been assigned a standard set of values but users are free to change these values if they so wish. The standard values are based on a variety of estimation or calibration procedures.

The coefficients in equations (1) and (2) are based on historical series for starts of "singles" (representing owner-type units) and "multiples" (representing renter-type units). In estimating natural values of starts and other variables in the system polynomial trend functions were fitted to data for the 25-year period 1963-87. Polynomials of degrees 1, 2, 3, and 4 were fitted and the results from each were taken into account in setting final

natural levels, growth rates, and cyclical parameters. Proportionate differences from trend were taken as a reflection of "cyclical" variations (as well as "irregular" fluctuations). More or less typical cycle lengths, amplitudes, and lead-lag characteristics were inferred from an examination of the 25-year series of proportionate differences.

The estimation and identification of cycles were based on seasonally adjusted quarterly data in the case of housing starts, personal income, the unemployment rate, the housing price index, and interest rates. However, for housing stocks and vacancies it was necessary to make do with annual data since quarterly data were not available. Values for the $\bar{\varphi}_h$ parameters in equation (10) were based on recent trend values of the vacancy series: $\bar{\varphi}_1 = .02$; $\bar{\varphi}_2 = .04$. Quarterly rates of discard or demolition, as required for the calculation of the $\bar{\beta}_{h0}$ parameters in equations (4) and (5), were simply assumed; .0025 was chosen as a standard value, so that $\bar{\beta}_{10} = \bar{\beta}_{20} = .9975$. Standard values of the parameters $\bar{\beta}_{hj}$ (j=1,2,3) were specified as follows: $\bar{\beta}_{11} = 0.57$, $\bar{\beta}_{12} = 0.43$, $\bar{\beta}_{13} = 0$; $\bar{\beta}_{21} = 0.2$, $\bar{\beta}_{22} = 0.8$, $\bar{\beta}_{23} = 0$. (These parameters represent the proportions of previous starts completed by the beginning of the current period.)

The logistic function representing the cumulative income distribution of renter households was estimated using 1982 renter data from the Statistics Canada Survey of Family Expenditures (FAMEX). (Income was defined as income of household "head" plus income of spouse, if spouse was present.) The fit was extremely good: the parameter values were calculated by fitting equation (11) and an R^2 value of .96 was obtained for this equation.

The parameter ϵ in equation (15) was assigned a value of 1. This parameter determines the responsiveness of renter incomes to changes in overall average personal income, and hence the responsiveness of the $q-\bar{q}$ variables in the starts equations.

The $\bar{\alpha}_{13}$ parameter required for the calculation of α_{13} in equations (3) and (22) was given the value .012 consistent with the "observed" natural values of $\bar{S}_1$ and $\bar{Q}$. In the absence of a government program of the kind contemplated for purposes of the model, only α_{13} would be operative historically and an appropriate choice of value for this parameter can therefore be inferred. However, there is no basis in historical data for inferring values for the θ parameters of equation (22) and these are simply assumed (but subject to change by a user). Our assumed standard value is 1 for both θ_1 and θ_2. A standard value of 1 is assumed also for $\bar{\zeta}$ in calculating the ζ parameter of equation (23). $\bar{\alpha}_{14}$ and $\bar{\alpha}_{24}$, required for equation (3), are set equal to 0.5 and 0.1, respectively.

The η_0 and η_1 parameters of equation (24) were estimated in the first instance using 1982 FAMEX data for home owners whose homes were constructed in 1981 or 1982. Subsequent adjustments to the estimated parameter values were made in order to calibrate the model so as to generate more or less "reasonable" relationships between home value and income level at the lower and upper ends of the relevant home value range. (What is "reasonable" is not easy to judge, it may be noted, given that the model is supposed to represent all of Canada, including Metropolitan Toronto and other high-price urban housing areas, on the one hand, and rural areas and small towns across the country, on the other.) In practice, a user can specify two points on the linear function defined by equation

(24) and the computer program will then calculate the implied η_0 and η_1 values.

The λ_1 and λ_2 parameters of equation (27) are strictly a matter of choice; to our knowledge there are no data that would indicate appropriate values for these parameters. We have set both of them to unity as a standard value, implying no displacement of other funds in the application of cash grants and interest-free loans to new home purchases.

The parameter ξ in equation (28) represents the maximum ratio of conventional mortgage loan to home purchase price. We have set this equal to 0.9 as a standard value.

The discounting functions $\sum_1$, $\sum_2$, and $\sum_3$ are based on ρ, which is defined as the conventional mortgage lending rate converted to a quarterly rate. An overall amortization period of 25 years is assumed, so that T = 100. A maximum annual gross debt service ratio of 30 percent is assumed; hence $\hat{r} = 0.3$.

The minimum natural income for the purchase of a new home in the base period is set at $25,000. (As noted previously, the original base period for the model was 1982. However, for computer implementation we have found it convenient to shift the base period to the calendar year 1988. The $25,000 minimum therefore refers to 1988.) Obviously this is a somewhat arbitrary choice and a user may wish to change it. (Again the choice of a minimum value is complicated by the fact that the model is intended to represent some sort of "average" across the whole of the country.)

The parameters σ_1 and σ_2 represent "slippage" in applications for benefit by qualified persons under the government program. We

have set these parameters to 0.9 in both cases but a user may prefer (or wish to experiment with) other values.

The π parameters in equations (40) and (41) represent the numbers of person-quarters of employment generated per new housing unit. Direct, indirect, and induced effects are defined separately. We have given standard values to the π parameters on the basis of estimates compiled by Clayton Research Associates and provided to us by CMHC, together with our own supplementary calculations. The lag distribution of the effects (up to two quarters) is based on the assumed relationships between starts and completions embodied in the parameters $\bar{\beta}_{h1}$, $\bar{\beta}_{h2}$, and $\bar{\beta}_{h3}$ of equations (4) and (5).

The choices of the natural growth rates appearing in equations (44)-(46) is quite arbitrary. One could take recent historical values but since natural growth is not of much relevance for purposes of studying cyclical stability we have thought it appropriate to set all of the rates to zero as a standard value (i.e., $g(\bar{Y}) = g(\bar{L}) = g(\bar{S}) = 0$).

As noted above, "typical" properties of cycles in different series were inferred by inspection of the time series of proportionate differences from trend over the period 1963-87. Standard values of the d, ω, and p parameters of equations (47)-(50) were chosen in this way. The d and ω parameters may vary among the four equations but the standard value of 25 quarters assigned to p is the same for all of them.

A standard value for $\bar{R}$ was chosen on the basis of recent "observed" values of the selected trend function fitted to the R series. The value chosen is 9.0 percent. A standard value for $\bar{u}$ was chosen in the same way; it too is 9.0 percent.

APPENDIX A: SUMMARY FORM OF THE MODEL

STARTS, STOCKS, AND VACANCIES

$$s_{ht} = \alpha_{h4} s'_{ht} + \alpha_{h5}(s_{h,t-1}) \qquad (h=1,2)$$

$$s'_{1t} = \sum_{i=1}^{3} \alpha_{1it}(q_{it} - \bar{q}_{it})$$

$$s'_{2t} = -\zeta s'_{1t}$$

$$k_{ht} = \beta_{h0} k_{h,t-1} + \sum_{j=1}^{3} \beta_{hj} s_{h,t-j} \qquad (h=1,2)$$

$$v_{ht} = \gamma_{h0} v_{h,t-1} - \gamma_{h1}(s'_{h,t-1} - s_{h,t-1}) \qquad (h=1,2)$$

$$\varphi_{ht} = \left((1+v_{ht})/(1+k_{ht})\right)\bar{\varphi}_h \qquad (h=1,2)$$

RENTER INCOME DISTRIBUTION

$$F_t(\bar{y}) = \left[1+\exp(-\delta_0)(\bar{y}\cdot\bar{Y}_0/\bar{Y}_t)^{-\delta_1}\right]^{-1}$$

$$y_t = \bar{y}_t(Y_t/\bar{Y}_t)^{\epsilon}$$

POTENTIAL PURCHASERS

$$q_{0t} = F_t(z_{1t})$$

$$q_{it} = F_t(z_{i+1,t}) - F_t(z_{it}) \qquad (i=1,2)$$

$$q_{3t} = 1 - F_t(z_{3t})$$

$$\bar{q}_{0t} = F_t(\bar{z}_{1t}) = F_t(\bar{z}_{2t}) = F_t(\bar{z}_{3t})$$

$$\bar{q}_{it} = 0 \qquad (i=1,2)$$

$$\bar{q}_{3t} = 1 - F_t(\bar{z}_{3t})$$

PROPENSITIES TO PURCHASE

$$\alpha_{1it} = \alpha_{13}\left(1+n_{it}/MV(w)_{it}\right)^{\theta_1} \qquad (i=1,2)$$

HOME VALUE

$$\bar{w}_t = \eta_0\left(\bar{Y}_t/\bar{Y}_0\right) + \eta_1 \bar{y}_t$$

$$w_t = \bar{w}_t(P/\bar{P})_t$$

FINANCING OF HOME PURCHASE

$$w_t = a_t + b_t + \lambda_1 c_{1t} + \lambda_2 c_{2t}$$

$$b_t = \xi w_t - \lambda_1 c_{1t} - \lambda_2 c_{2t}$$

$$b_t = m_{1t}\sum_{1t} + m_{2t}\sum_{2t}$$

$$\sum_{1t} = \sum_{\tau=1}^{T'} (1+\rho_t-(\Delta\rho)_t)^{-\tau}$$

$$\sum_{2t} = \sum_{\tau=T'+1}^{T} (1+\rho_t)^{-\tau}$$

$$m_{2t} = b_t / \sum_{3t}$$

$$\sum_{3t} = \sum_{\tau=1}^{T} (1+\rho_t)^{-\tau}$$

$$r_t = 4m_{1t}/y_t$$

INCOME BOUNDARY VALUES

$$y_t^* = A_t/B_t$$

$$A_t = \xi\eta_0(\bar{Y}_t/\bar{Y}_0)(P/\bar{P})_t - (\lambda_1 c_{1t}+\lambda_2 c_{2t})$$

$$B_t = (1/D_t)(\hat{r}/4)(Y/\bar{Y})_t^{\epsilon} - \xi\eta_1(P/\bar{P})_t$$

$$D_t = \left(1-\sum_{2t}/\sum_{3t}\right)/\sum_{1t}$$

$z_{1t} = y_t^*$ when all variables have actual values

$z_{2t} = y_t^*$ when $c_{1t} = c_{2t} = (\Delta\rho)_t = 0$

$\bar{z}_{3t} = y_t^*$ when $c_{1t} = c_{2t} = (\Delta\rho)_t = 0$, $(P/\bar{P})_t = (Y/\bar{Y})_t = 1$, and $\rho_t = \bar{\rho}$

Note: $\bar{z}_{1t} = \bar{z}_{2t} = \bar{z}_{3t}$

$$\hat{y}_t = \left[\hat{w}_t-\eta_0(\bar{Y}_t/\bar{Y}_0)(P/\bar{P})_t\right]/\eta_1\left(P/\bar{P}\right)_t$$

$z_{3t} = \hat{y}_t$ when all variables have actual values

Final adjustment: all z and $\bar{z}$ values are increased by the amount $\bar{Z}_t - \bar{z}_{2t}$, where $\bar{Z}_t = \bar{Z}_0(\bar{Y}_t/\bar{Y}_0)$

VALUE OF PROGRAM BENEFIT

$$n_{it} = c_{1t} + c_{2t}\left(1-(1+\rho_t')^{-T''}\right) + \left(MV(m_2-m_1)_{it}\right)\sum_{4t} \qquad (i=1,2)$$

$$\sum_{4t} = \sum_{\tau=1}^{T'} (1+\rho_t')^{-\tau}$$

$$m_2 - m_1 = \left[\xi(P/\bar{P})_t\left(\eta_0(\bar{Y}_t/\bar{Y}_0)+\eta_1\bar{y}_t\right) - \left(\lambda_1 c_{1t}+\lambda_2 c_{2t}\right)\right]M_t$$

$$M_t = [1-(\sum_{3t}-\sum_{2t})/\sum_{1t}]/\sum_{3t}$$

PROGRAM PARTICIPATION

$$N_{1t} = \sigma_1(\alpha_{11t}q_{1t}/\sum_{i=1}^{3}\alpha_{1it}q_{it})(1+s_{1t})\bar{S}_{1t}$$

$$N_{2t} = \sigma_2(\alpha_{12t}q_{2t}/\sum_{i=1}^{3}\alpha_{1it}q_{it})(1+s_{1t})\bar{S}_{1t}$$

$$AN_t = AN_{t-1} + N_t$$

PROGRAM COST

$$C_t = \sum_{i=1}^{2}N_{it}[c_{1t}+c_{2t}(1-(1+\rho_t'')^{-T''}) + (MV(m_2-m_1)_{it})\sum_{5t}]$$

$$\sum_{5t} = \sum_{\tau=1}^{T'}(1+\rho_t'')^{-\tau}$$

EMPLOYMENT CREATION

$$(\Delta E)_{ht} = \sum_{j=0}^{2}\pi_{hj}(s_{h,t-j} - s^*_{h,t-j})\bar{S}_{h,t-j} \qquad (h=1,2)$$

$$\pi_{hj} = \pi_{hj}^{(1)} + \pi_{hj}^{(2)} + \pi_{hj}^{(3)} \qquad (h=1,2;\ j=0,1,2)$$

$$(\Delta E)_t = (\Delta E)_{1t} + (\Delta E)_{2t}$$

$$u_t = u_t^* - (\Delta E)_t/\bar{L}_t$$

NATURAL GROWTH RATES

$$\bar{Y}_t = \bar{Y}_{t-1}(1+g(\bar{Y}))$$

$$\bar{L}_t = \bar{L}_{t-1}(1+g(\bar{L}))$$

$$\bar{S}_{ht} = \bar{S}_{ht-1}(1+g(\bar{S})) \qquad (h=1,2)$$

DETERMINATION OF CYCLES

$$(Y/\bar{Y})_t - 1 = d_1\sin[(2\pi/p)(t-\omega_1)]$$

$$(P/\bar{P})_t - 1 = d_2\sin[(2\pi/p)(t-\omega_2)]$$

$$(R-\bar{R})_t = d_3\sin[(2\pi/p)(t-\omega_3)]$$

$$(u^*-\bar{u})_t = d_4\sin[(2\pi/p)(t-\omega_4)]$$

QUARTERLY INTEREST RATES

$$\rho_t = (1+R_t/2)^{1/2} - 1 \qquad \text{(similar equations for } \rho', R', \rho'', R'')$$

$$\bar{\rho} = \left(1+\bar{R}/2\right)^{1/2} - 1$$

$$(\Delta\rho)_t = \left(1+R_t/2\right)^{1/2} - \left(1+\left(R_t-(\Delta R)_t\right)/2\right)^{1/2}$$

APPENDIX B: DEFINITIONS

Latin symbols

a	down payment associated with home purchase
b	amount borrowed from commercial source for home purchase
C	total cost of benefits paid under government program
c_1	cash grant under government program
c_2	interest-free loan under government program
d	parameter representing amplitude of cycle in sine function
ΔE	employment creation associated with government program, in person quarters
F(.)	cumulative natural income distribution function for renter households (proportion of households at or below value of argument)
g	natural growth rate, also function indicating natural growth rate
h	subscript indicating housing type (h = 1 for owner-type; h = 2 for renter-type)
i	subscript indicating purchaser category (i=0,1,2,3)
j	subscript indicating time lag, in quarters
k	stock (number) of housing units as proportionate difference from natural level at beginning of period
$\bar{L}$	natural labour force
MV(.)	mean value function (mean value of argument)
m_1	quarterly mortgage payment with government interest-rate subsidy program in effect
m_2	quarterly mortgage payment without government interest-rate subsidy program in effect
N	number of households participating in government program
n	present value of program benefits to a participating household
P	housing price index (owner-type housing)

$\bar{P}$	natural value of P
p	parameter representing period or length of cycle in sine function
q	proportion of renter households in given potential purchaser category
$\bar{q}$	natural proportion of renter households in given potential purchaser category
R	annual quoted mortgage interest rate
$\bar{R}$	natural value of R
ΔR	reduction of R under government program
r	gross debt service ratio
$\hat{r}$	maximum gross debt service ratio
S	housing starts
$\bar{S}$	natural level of housing starts
s	housing starts as proportionate difference from natural level
s′	demand for housing starts as proportionate difference from natural level
s^*	value of s in complete absence of government program
T	amortization period for mortgage (number of quarters)
T′	length of period of government interest rate subsidy (number of quarters)
T"	length of period of government interest-free loan (number of quarters)
t	index indicating time, in quarters (t = 1,2,...)
v	vacancies as proportionate difference from natural level at beginning of period
w	purchase value of home
$\bar{w}$	natural level of w
$\hat{w}$	maximum home value for program eligibility
Y	average personal income per household
$\bar{Y}$	natural level of Y

y	income of an individual renter household
$\bar{y}$	natural level of y
y^*	minimum $\bar{y}$ to allow home purchase
$\hat{y}$	maximum $\bar{y}$ for program eligibility
u	unemployment rate
u^*	unemployment rate without government program
$\bar{Z}$	independently set minimum value of income for government program eligibility
z	point on natural income distribution when system is in actual current state
$\bar{z}$	point on natural income distribution when system is in natural state

Greek symbols

α	parameter in housing starts or demand equation
β	parameter in capital stock equation
γ	parameter in vacancy equation
δ	parameter in natural income distribution function
ϵ	parameter in household income equation
ζ	parameter relating demand for rental-type starts to that for owner-type starts
η	parameter in home value equation
θ	parameter in propensity to purchase equation
λ	parameter associated with government housing program
ξ	mortgage loan as fraction of home purchase value
π	employment multiplier (superscript 1 indicates direct, 2 indirect, and 3 induced); also, value 3.14... in cycle function
ρ	mortgage interest rate (quarterly rate)
$\bar{\rho}$	natural value of ρ

ρ'	rate of interest at which households discount future returns or costs
ρ''	rate of interest at which government discounts future returns or costs
$\Delta\rho$	reduction of ρ under program
$\sum$	symbol representing present value calculation $(\sum_1, \ldots, \sum_5)$
σ	proportion of eligible home purchasers who participate in government program
τ	index indicating future time (in quarters) for present-value calculations
φ	vacancy rate (vacancies as proportion of stock)
$\bar{\varphi}$	natural vacancy rate
ω	parameter representing displacement or time shift of cycle in sine function